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Synthesis of Time-Variant Optimal Control with Nonquadratic Criteria

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Under the assumption of convexity of nonquadratic time-variant criteria for a linear time-variant control system, it is proved in this paper that the closed-loop synthesis of the optimal control is given by a nonlinear feedback

$$u(t) = -R^{-1}(t)B^*(t)P(t, x(t)),$$

in which $P(t, x)$ is the normal solution of a quasi-Riccati operator equation. It is also shown that the nonlinear feedback operator $P(t, x)$ can be explicitly expressed by solutions of the associated nonlinear algebraic equation and nonlinear integral equations respectively in three cases corresponding to Mayer problems, Lagrange problems, and Bolza problems. © 1997 Academic Press

1. INTRODUCTION

It is well known, cf. [1, 4], that time optimal control and quadratic optimal control for linear systems can be synthesized in closed-loop forms. However, how to implement an optimal control for a time-variant linear system with a nonquadratic time-variant cost function by a real-time state feedback is an important problem in practical applications but remains theoretically open.

This paper deals with the optimal control problem for a linear system

$$\frac{dx}{dt} = A(t)x + B(t)u(t), \qquad x(0) = x_0, \tag{1.1}$$

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with respect to the nonquadratic cost function

$$\min_{u(\cdot)\in \mathcal{U}} \left\{ J(u) = M(x(T)) + \int_0^T \left[Q(t, x(t)) + \frac{1}{2}\langle R(t)u(t), u(t)\rangle \right] dt \right\}. \tag{1.2}$$

This problem will be referred to as (NQR).

Assume that the state functions $x(t)$ and the initial data x_0 take values in $X = \mathbf{R}^n$ and that the control functions $u(t)$ take values in $U = \mathbf{R}^m$. Let $\langle \cdot, \cdot \rangle$ denote inner products. T is a fixed finite positive number. Denote function spaces by $\mathcal{X} = L^2(0, T; X)$, $\mathcal{X}_c = C([0, T]; X)$, and $\mathcal{U} = L^2(0, T; U)$. Whenever the definition interval of functions is $[t, T]$, the corresponding function spaces will be denoted by $\mathcal{X}[t, T]$, $\mathcal{X}_c[t, T]$, and $\mathcal{U}[t, T]$, respectively.

In this paper we make the following assumptions:

(A1) $A(t)$ and $B(t)$ are respectively $n \times n$ and $n \times m$ continuous real matrix functions.

(A2) $M(x)$: $X \to \mathbf{R}$ is a C^2 and convex function.

(A3) $Q(t, x)$: $[0, T] \times X \to \mathbf{R}$ and its partial derivatives $Q_x(t, x)$ and $Q_{xx}(t, x)$ are continuous in (t, x). For each $t \in [0, T]$, $Q(t, x)$ is convex in x.

(A4) $R(t)$ is an $m \times m$ continuous and symmetric real matrix function, and there is a constant $\delta > 0$ such that $R(t) \geq \delta I$ for $t \in [0, T]$.

Let $E(s, t)$ denote the fundamental matrix of the linear system (1.1). Then, the solution $x(t)$ of (1.1) is expressed by the formula

$$x(t) = E(t, 0)x_0 + \int_0^t E(t, s)B(s)u(s)\, ds, \qquad t \in [0, T]. \tag{1.3}$$

The aim of this paper is to prove that there exists a closed-loop optimal control $u(t) = V(t, x(t))$ for (NQR) and to provide approaches for finding the feedback operators by solving some classes of equations.

Let $(\text{NQR})_1$ and $(\text{NQR})_2$ refer to the special cases of (NQR) with $Q(t, x) \equiv 0$ and $M(x) \equiv 0$, respectively.

In this paper, a mapping's derivative always means its Fréchet derivative, denoted by a superscript $'$ or by a preceded D. Subscripts will denote partial derivatives or some parameters. Superscript * will represent adjoint operators and transposed matrices.

All the concepts and results of nonlinear analysis used in this paper can be found in [2]. In particular, we shall often use the following relation between a Fréchet differentiable mapping f defined on a Banach space and its derivative Df,

$$f(x+h)-f(x)=\int_0^1 Df(x+sh)h\,ds. \tag{1.4}$$

2. EXISTENCE, UNIQUENESS, AND OPEN-LOOP EQUATIONS

THEOREM 2.1. *For any given $x_0 \in X$, there exists a unique optimal control for (NQR).*

Proof. By the property of a C^1 convex function $f(\cdot)$ defined on a Hilbert space H, i.e.,

$$f(\xi) \geq f(0) + \langle f'(0), \xi \rangle, \qquad \forall \xi \in H,$$

and the assumptions (A1)–(A4), there exist constants $\beta > 0$ and $\alpha(x_0) \in \mathbf{R}$ depending continuously on x_0, such that

$$\begin{aligned}\tfrac{1}{2}\delta\|u\|_U^2 &\leq \tfrac{1}{2}\int_0^T \langle R(t)u(t), u(t)\rangle\, dt \\ &= J(u;x_0) - M\left(E(T,0)x_0 + \int_0^T E(T,s)B(s)u(s)\,ds\right) \\ &\quad - \int_0^T \left[Q\left(t, E(t,0)x_0 + \int_0^t E(t,s)B(s)u(s)\,ds\right)\right] dt \\ &\leq J(u;x_0) + \alpha(x_0) + \beta\|u\|_U. \end{aligned} \tag{2.1}$$

This leads to

$$0 \leq \frac{\delta}{2}\left(\|u\|_U - \frac{\beta}{\delta}\right)^2 \leq J(u;x_0) + \left[\alpha(x_0) + \frac{\beta^2}{2\delta}\right], \qquad \forall u \in U. \tag{2.2}$$

Inequality (2.2) indicates that $\inf_{u\in U} J(u;x_0)$ is finite for a given x_0 and that a minimizing sequence is uniformly bounded so that a weakly convergent subsequence exists in U. On the other hand, from the convexity we have

$$\begin{aligned}\theta J(u) + (1-\theta)J(v) &\geq J(\theta u + (1-\theta)v) \\ &\quad + \frac{1}{2}\theta(1-\theta)\int_0^T \langle R(t)(u-v), u-v\rangle\, dt.\end{aligned} \tag{2.3}$$

Thus $J(u)$ is a strictly convex and strongly continuous functional on $\boldsymbol{U}$, hence weakly lower semi-continuous. Therefore, by the basic theorem in variational analysis, for any given x_0, (NQR) has a unique optimal control. ▮

THEOREM 2.2. *If $u(\cdot)$ is the optimal control and $x(\cdot)$ is the corresponding optimal trajectory, then $\{u, x\}$ satisfies the open-loop equation,*

$$u(t) = -R^{-1}(t)B^*(t)\left\{E^*(T,t)M'(x(T)) + \int_t^T E^*(s,t)Q'_x(s,x(s))\,ds\right\}, \qquad t \in [0,T] \quad (2.4)$$

This is a consequence of the Pontryagin Maximum Principle applied to this problem (NQR). The details are omitted.

COROLLARY 2.3. *Let $t_0 \in [0,T]$. Then the following optimal control problem* $(\mathrm{NQR})_{t_0}$,

$$x(t) = E(t,t_0)x_0 + \int_{t_0}^t E(t,s)B(s)u(s)\,ds, \qquad t \in [t_0,T], \quad (2.5)$$

$$\min_{u \in \boldsymbol{U}[t_0,T]} \left\{J_{t_0}(u) = M(x(T)) + \int_{t_0}^T \left[Q(t,x(t)) + \frac{1}{2}\langle R(t)u(t), u(t)\rangle\right] dt\right\} \quad (2.6)$$

has a unique optimal control for any given x_0, and the optimal process $\{u(\cdot), x(\cdot)\}$ of $(\mathrm{NQR})_{t_0}$ *satisfies* (2.4) *for $t \in [t_0,T]$.*

3. QUASI-RICCATI EQUATION AND CLOSED-LOOP SYNTHESIS

In this section we study a quasi-Riccati equation associated with (NQR),

$$\begin{cases} P_t(t,x) + P_x(t,x)A(t)x + A^*(t)P(t,x) \\ \quad -P_x(t,x)B(t)R^{-1}(t)B^*(t)P(t,x) + Q_x(t,x) = 0, \\ \qquad\qquad (t,x) \in [0,T] \times X, \\ P(T,x) = M'(x), \qquad x \in X, \end{cases} \quad (3.1)$$

and its relationship with the closed-loop synthesis of the concerned optimal control problem.

DEFINITION 3.1. A continuous mapping $P(t, x)$: $[0, T] \times X \to X$ is called a normal solution of the quasi-Riccati equation (3.1), if it satisfies the following conditions:

(1) $P(t, x)$ is continuously differentiable in t and x, respectively, and it satisfies Eq. (3.1).

(2) For $0 \le t \le T$, $P(t, \cdot)$: $X \to X$ is gradient operator (see [2, p. 93]).

(3) The Cauchy problem

$$\frac{dx}{dt} = A(t)x - B(t)R^{-1}(t)B^*(t)P(t, x), \qquad x(0) = x_0, \quad (3.2)$$

has a global solution $x(\cdot) \in \mathsf{X}_c$ on $[0, T]$ for any given $x_0 \in \mathsf{X}$.

Let $P(t, x)$ be a normal solution of (3.1). According to the definition of gradient operators, for each $t \in [0, T]$, the anti-derivatives of $P(t, x)$ are nonlinear functionals $\Phi(t, x): [0, T] \times X \to \mathbf{R}$ such that

$$\Phi_x(t, x) = P(t, x), \qquad (t, x) \in [0, T] \times X. \quad (3.3)$$

These $\Phi(t, x)$ may well differ from each other by constants. To fix ideas without loss of generality, we set

$$\Phi(t, 0) = M(0), \qquad 0 \le t \le T. \quad (3.4)$$

LEMMA 3.2. *Let $P(t, x)$ be a normal solution of the quasi-Riccati equation* (3.1) *and $\Phi(t, x)$ be its anti-derivative satisfying* (3.4). *Let $x(t)$ be a trajectory corresponding to any initial state x_0 and any admissible control $u(\cdot) \in \mathsf{U}$. Then $\Phi(t, x(t))$ is an absolutely continuous function on* $[0, T]$.

Proof. Obviously, the state function $x(t)$ is absolutely continuous. By (1.4), (3.3), and (3.4), we have

$$\Phi(t, x(t)) = \int_0^1 \langle P(t, sx(t)), x(t) \rangle \, ds + M(0), \qquad t \in [0, t]. \quad (3.5)$$

Let $\Pi = \mathrm{C1.Conv}\{sx(t) | 0 \le s \le 1, 0 \le t \le T\}$, where $x(t)$ is a trajectory mentioned above. By Definition 3.1, $P(t, x)$, $P_t(t, x)$, and $P_x(t, x)$ are all uniformly bounded in their norms on the convex compact set $[0, T] \times \Pi$. From the mean value theorem [2], it follows that $P(t, x)$ satisfies a uniform Lipschitz condition with respect to $(t, x) \in [0, T] \times \Pi$. These facts imply that the $\Phi(t, x(t))$ shown by (3.5) is absolutely continuous. ∎

LEMMA 3.3. *The assumptions are the same as in Lemma 3.2. Then,*

$$\begin{aligned}\frac{d}{dt}\Phi(t, x(t))& \\ \doteq -Q(t, x(t)) &+ Q(t,0) + \langle B(t)u(t), P(t, x(t))\rangle \\ &+ \int_0^1 \langle P_x(t, sx(t))B(t)R^{-1}(t)B^*(t)P(t, sx(t)), x(t)\rangle\, ds. \end{aligned} \tag{3.6}$$

Proof. From Lemma 3.2 and the fact that the integrand in (3.5) satisfies a uniform Lipschitz condition in $t \in [0, T]$ and the Lipschitz constant can be taken as independent of the integral variable $s \in [0, 1]$, we differentiate both sides of (3.5) in t and obtain, according to the differentiation theorem for Lebesgue integrals with parameters, that

$$\begin{aligned}\frac{d}{dt}\Phi(t, x(t))& \\ \doteq \int_0^1 \big[&\langle P_t(t, sx(t)), x(t)\rangle + \langle P_x(t, sx(t))\dot{x}(t), x(t)\rangle s \\ &+ \langle P(t, sx(t)), \dot{x}(t)\rangle\big]\, ds \\ \doteq \int_0^1 \big[&\langle P_t(t, sx(t)), x(t)\rangle + \langle P_x(t, sx(t))A(t)sx(t), x(t)\rangle \\ &+ \langle A^*(t)P(t, sx(t)), x(t)\rangle\big]\, ds \\ &+ \left\langle B(t)u(t), \int_0^1 P(t, sx(t))\, ds\right\rangle \\ &+ \int_0^1 \langle P_x(t, sx(t))B(t)u(t), x(t)\rangle s\, ds \\ = -\int_0^1 &\langle Q_x(t, sx(t)), x(t)\rangle\, ds \\ &+ \int_0^1 \langle P_x(t, sx(t))B(t)R^{-1}(t)B^*(t)P(t, sx(t)), x(t)\rangle\, ds \\ &+ \left\langle B(t)u(t), \int_0^1 P(t, sx(t))\, ds\right\rangle \\ &+ \int_0^1 \langle B(t)u(t), P_x(t, sx(t))x(t)\rangle s\, ds. \end{aligned} \tag{3.7}$$

Here $P_x(t, x) \in L(X)$ is self-adjoint because $P(t, x)$ is a gradient operator in x. Now integrate the last term in (3.7) by parts and obtain

$$\int_0^1 \langle B(t)u(t), P_x(t, sx(t))x(t)\rangle s\, ds$$

$$= \int_0^1 \langle B(t)u(t), P_x(t, sx(t))x(t)\rangle\, ds$$

$$- \int_0^1 \int_0^s \langle B(t)u(t), P_x(t, \eta x(t))x(t)\rangle\, d\eta\, ds.$$

By (1.4), we have

$$\int_0^s P_x(t, \eta x(t))x(t)\, d\eta + \int_0^1 P_x(t, \sigma sx(t))sx(t)\, d\sigma$$

$$= P(t, sx(t)) - P(t, 0).$$

Let these results be substituted into (3.7), we obtain (3.6). ∎

THEOREM 3.4. *Let $P(t, x)$ be a normal solution of the quasi-Riccati equation* (3.1). *Then, for any given $x_0 \in X$, the feedback control*

$$u(t) = -R^{-1}(t)B^*(t)P(t, x(t)), \qquad t \in [0, T], \tag{3.8}$$

must be the closed-loop optimal control of (NQR), *where $x(t)$ is the value of $x(\cdot)$ starting from x_0 when this feedback control $u(\cdot)$ is used.*

Proof. Let $\Phi(t, x)$ be the anti-derivative of $P(t, x)$ and let it satisfy (3.4). Since the function

$$\theta(t, x) = \frac{1}{2}\langle R^{-1}(t)B^*(t)P(t, x), B^*(t)P(t, x)\rangle \tag{3.9}$$

has partial derivative $\theta_x(t, x) = P_x(t, x)B(t)R^{-1}(t)B^*(t)P(t, x)$, from Lemma 3.3 we know that for any control $u(\cdot) \in U$ and its corresponding $x(\cdot)$,

$$\frac{d}{dt}\Phi(t, x(t)) + Q(t, x(t)) + \tfrac{1}{2}\langle R(t)u(t), u(t)\rangle$$

$$\doteq \frac{1}{2}\langle R(t)\left[u(t) + R^{-1}(t)B^*(t)P(t, x(t))\right],$$

$$u(t) + R^1(t)B^*(t)P(t, x(t))\rangle$$

$$+ Q(t, 0) - \theta(t, 0). \tag{3.10}$$

Integrate (3.10) for $t \in [0,T]$. On account of the absolute continuity of $\Phi(t, x(t))$, the equality $\Phi(T,x) = M(x)$, $x \in X$, which is implied by $\Phi_x(T,x) = P(T,x) = M'(x)$ and (3.4), we obtain

$$\begin{aligned} J(u) &= M(x(T)) + \int_0^T \left[Q(t,x(t)) + \frac{1}{2}\langle R(t)u(t), u(t)\rangle \right] dt \\ &= \Phi(0,x_0) + \int_0^T \left[\frac{d}{dt}\Phi(t,x(t)) + Q(t,x(t)) \right. \\ &\qquad\qquad\qquad \left. + \frac{1}{2}\langle R(t)u(t), u(t)\rangle \right] dt \\ &= \left\{ \Phi(0,x_0) + \int_0^T [Q(t,0) - \theta(t,0)]\, dt \right\} \\ &\quad + \frac{1}{2}\int_0^T \Big\langle R(t)\big[u(t) + R^{-1}(t)B^*(t)P(t,x(t))\big], \\ &\qquad\qquad\qquad u(t) + R^{-1}(t)B^*(t)P(t,x(t))\Big\rangle\, dt \\ &\ge \rho(x_0), \end{aligned} \tag{3.11}$$

where $\rho(x_0) = \Phi(0,x_0) + \int_0^T [Q(t,0) - \theta(t,0)]\, dt$ is a constant only depending on x_0. It is obvious that the feedback control (3.8) will make $J(u)$ attain its minimum $\rho(x_0)$ for any given x_0. In view of Definition 3.1, for any given x_0, the Cauchy problem (3.2) possesses a global solution $x(\cdot) \in \mathcal{X}_c$. Moreover the properties of $P(t,x)$ assure that the solution of (3.2) is unique. Thus the feedback control (3.8) is admissible in $\mathcal{U}$. Therefore it must be optimal. ∎

Since the quasi-Riccati equation (3.1) is a time dependent nonlinear partial differential equation for a nonlinear operator $P(t,x)$, we shall further investigate how to reduce its solution to a class of equations which can be solved or approximately solved via numerical algorithms for three cases $(\mathrm{NQR})_1$, $(\mathrm{NQR})_2$, and (NQR) respectively.

4. SYNTHESIS EQUATION FOR $(\mathrm{NQR})_1$

In this section we consider $(\mathrm{NQR})_1$ with $Q(t,x) \equiv 0$, whose synthesis is associated with the equation

$$y + \Lambda(t)M'(y) = E(T,t)x, \qquad x \in X,\ t \in [0,T], \tag{4.1}$$

where

$$\Lambda(t) = \int_t^T E(T,s)B(s)R^{-1}(s)B^*(s)E^*(T,s)\,ds \in \mathbf{L}(X).$$

For each $t \in [0,T]$, (4.1) is a nonlinear algebraic equation in y.

THEOREM 4.1. *For any given $(t,x) \in [0,T] \times X$, there exists a unique solution $y \in X$ of Eq. (4.1), denoted by mapping $y = H(t,x)$. $H(t,x)$ is continuously differentiable in (t,x), and*

$$H_t(t,x) = (I + \Lambda(t)M''(H(t,x)))^{-1}E(T,t)[-A(t)x + B(t)R^{-1}(t)B^*(t)E^*(T,t)M'(H(t,x))], \quad (4.2)$$

$$H_x(t,x) = (I + \Lambda(t)M''(H(t,x)))^{-1}E(T,t). \quad (4.3)$$

Proof. We first prove existence. Let $t \in [0,T]$ and $x \in X$ be arbitrarily fixed. By Corollary 2.3, for $(\mathrm{NQR})_t$ with $Q(t,x) \equiv 0$, there is a unique optimal process $\{\hat{u}(\cdot), \hat{x}(\cdot)\}$ with the initial state x, and (2.4) implies that

$$\hat{x}(s) = E(s,t) - \int_t^s E(s,\sigma)B(\sigma)R^{-1}(\sigma)B^*(\sigma)E^*(T,\sigma)M'(\hat{x}(T))\,d\sigma, \quad s \in [t,T]. \quad (4.4)$$

Equation (4.4) shows that $y = \hat{x}(T)$ is a solution of Eq. (4.1).

We now prove uniqueness and differentiability. Define a mapping

$$C(t,x,y) = y + \Lambda(t)M'(y) - E(T,t)x. \quad (4.5)$$

The convexity of $M(\cdot)$ implies that $M''(y) \in \mathbf{L}(X)$ is self-adjoint and nonnegative. Hence $C_y(t,x,y) = I + \Lambda(t)M''(y)$ is boundedly invertible and

$$C_y(t,x,y)^{-1} = I - \Lambda(t)\sqrt{M''(y)}\left[I + \sqrt{M''(y)}\,\Lambda(t)\sqrt{M''(y)}\right]^{-1}\sqrt{M''(y)}. \quad (4.6)$$

By the implicit function theorem, cf. [2], we conclude that the solution of Eq. (4.1) is unique and differentiable in parameters (t, x). Moreover,

$$H_t(t,x) = -\left[C_y(t,x,H(t,x))^{-1} C_t(t,x,H(t,x))\right],$$
$$H_x(t,x) = -\left[C_y(t,x,H(t,x))^{-1} C_x(t,x,H(t,x))\right].$$

Substitute the expressions C_y, C_t, and C_x into the above equalities to obtain (4.2) and (4.3). ∎

THEOREM 4.2. *Let* $H(t,x)$ *be the solution mapping of* (4.1). *Then,*

$$P(t,x) = E^*(T,t)M'(H(t,x)) \colon [0,T] \times X \to X \tag{4.7}$$

is a normal solution of the quasi-Riccati equation (3.1) *associated with* $(\mathrm{NQR})_1$.

Proof. It is obvious that $P(t,x)$ shown by (4.7) satisfies $P(T,x) = M'(x)$. Because of (4.2) and (4.3), and

$$\begin{aligned} P_t(t,x) &= -A^*(t)P(t,x) + E^*(T,t)M''(H(t,x))H_t(t,x), \\ P_x(t,x) &= E^*(T,t)M''(H(t,x))H_x(t,x), \end{aligned} \tag{4.8}$$

it can be verified that this $P(t,x)$ satisfies (3.1) with $Q(t,x) \equiv 0$.

Next we prove that $P(t, \cdot)$ is a gradient operator for each $t \in [0,T]$. By Theorem 2.5.2 in [2], we need only prove that $P_x(t,x) \in \mathcal{L}(X)$ is self-adjoint. The latter is shown by

$$\begin{aligned} P_x(t,x) &= E^*(T,t)M''(H(t,x))(I + \Lambda(t)M''(H(t,x)))^{-1}E(T,t) \\ &= E^*(T,t)\sqrt{M''(H(t,x))} \\ &\quad \times \left\{I + \sqrt{M''(H(t,x))}\,\Lambda(t)\sqrt{M''(H(t,x))}\right\}^{-1} \\ &\quad \times \sqrt{M''(H(t,x))}\,E(T,t). \end{aligned} \tag{4.9}$$

Finally we prove the existence of a global solution to the Cauchy problem (3.2). In fact, in view of the uniqueness just proved, the optimal trajectory $x(\cdot)$ of $(\mathrm{NQR})_1$ with the initial state x_0 satisfies

$$x(t) = H(t, x(t)), \qquad t \in [0,T]. \tag{4.10}$$

By Theorem 2.2 and (4.10), we see that this $x(\cdot) \in \mathcal{X}_c$ is a global solution of (3.2). ∎

5. SYNTHESIS EQUATION FOR $(\mathrm{NQR})_2$

In this section we consider $(\mathrm{NQR})_2$ with $M(x) \equiv 0$, which is connected with the synthesis equation

$$y(s) + \int_t^T K_t(s, \sigma) Q_x(\sigma, y(\sigma))\, d\sigma = E(s, t)x, \qquad s \in [t, T], \quad (5.1)$$

for any given $(t, x) \in [0, T] \times X$, where

$$K_t(s, \sigma) = \int_t^{\min(s, \sigma)} E(s, \eta) B(\eta) R^{-1}(\eta) B^*(\eta) E^*(\sigma, \eta)\, d\eta \in \mathsf{L}(X). \quad (5.2)$$

For any given $(t, x) \in [0, T] \times X$, (5.1) is a nonlinear Fredholm integral equation. We denote $\Omega = \{(s, t, x) | 0 \le t \le s \le T,\ x \in X\}$.

LEMMA 5.1. *Define an operator* $\Gamma_t \in \mathsf{L}(\mathsf{U}[t, T];\ \mathsf{X}[t, T])$ *by*

$$(\Gamma_t v)(s) = \int_t^s E(s, \eta) B(\eta) \sqrt{R^{-1}(\eta)}\, v(\eta)\, d\eta, \qquad s \in [t, T]. \quad (5.3)$$

Then the following statements hold:

(1) *Let* $N \in \mathsf{L}(\mathsf{X}[t, T])$ *be any nonnegative self-adjoint operator. Then* $I + \Gamma_t \Gamma_t^* N \in \mathsf{L}(\mathsf{X}[t, T])$ *is boundedly invertible, and*

$$(I + \Gamma_t \Gamma_t^* N)^{-1} = I - \Gamma_t \Gamma_t^* \sqrt{N} \left(I + \sqrt{N} \Gamma_t \Gamma_t^* \sqrt{N}\right)^{-1} \sqrt{N}. \quad (5.4)$$

(2) *If* $N \in \mathsf{L}(\mathsf{X}[t, T])$ *is defined by*

$$(N\phi)(s) = Y(s)\phi(s), \qquad s \in [t, T], \quad (5.5)$$

where $Y(s) \in \mathsf{L}(X)$ *is strongly continuous in* s *and nonnegative self-adjoint on* X, *then* (1) *holds, and furthermore,* $I + \Gamma_t \Gamma_t^* N \in \mathsf{L}(\mathsf{X}_c[t, T])$ *is boundedly invertible and its inverse operator is also given by* (5.4).

The proof is straightforward and omitted.

LEMMA 5.2. *The synthesis equation* (5.1) *has a unique solution* $y(s; t, x)$ *such that* $y(\cdot; t, x) \in \mathsf{X}_c[t, T]$.

Proof. Let $(t, x) \in [0, T] \times X$ be arbitrary and fixed. By Corollary 2.3, for $(\mathrm{NQR})_t$ with $M(x) \equiv 0$, there is a unique optimal process $\{\hat{u}(\cdot), \hat{x}(\cdot)\}$ with the initial state x. Equation (2.4) implies that $y(s; t, x) = \hat{x}(s)$, $s \in$

$[t, T]$, is exactly a solution of Eq. (5.1) in $X_c[t, T]$,

$$\hat{x}(s) = E(s,t)x - \int_t^s E(s,\eta)B(\eta)R^{-1}(\eta)B^*(\eta) \times \int_\eta^T E^*(\sigma,\eta)Q_x(\sigma,\hat{x}(\sigma))\,d\sigma\,d\eta$$

$$= E(s,t)x - \int_t^T K_t(s,\sigma)Q_x(\sigma,\hat{x}(\sigma))\,d\sigma, \tag{5.6}$$

where the last equality is obtained by interchange of the integration order.

Define a mapping F: $X_c[t, T] \times [0, T] \times X \to X_c[t, T]$ by

$$F(y(\cdot),t,x)(s) = y(s) + \int_t^T K_t(s,\sigma)Q_x(\sigma,y(\sigma))\,d\sigma - E(s,t)x, \quad s \in [t,T]. \tag{5.7}$$

For each fixed $(t, x) \in [0, T] \times X$, we have

$$D_yF(y(\cdot),t,x) = I + \Gamma_t\Gamma_t^* Q_{xx}(\cdot, y(\cdot)), \tag{5.8}$$

where $(Q_{xx}(\cdot, y(\cdot))z)(s) = Q_{xx}(s, y(s))z(s)$, $s \in [t, T]$. By the assumption (A3) and Lemma 5.1, we know that for each $y(\cdot) \in X_c[t, T]$, $D_yF(y(\cdot), t, x) \in L(X_c[t, T])$ is boundedly invertible. Then the uniqueness is proved by the implicit function theorem. ∎

Denote the unique solution of (5.1) by a mapping $G(s, t, x)$: $\Omega \to X$,

$$G(s,t,x) = y(s;t,x). \tag{5.9}$$

LEMMA 5.3. *For any fixed bounded subset Σ in X, the solution mapping $G(s, t, x)$ of* (5.1) *is uniformly bounded in norm on $\Omega_\Sigma = \{(s, t, x) | 0 \le t \le s \le T,\ x \in \Sigma\}$.*

Proof. Let $J_{t_0}^*(x_0) = \min_{u \in U[t_0, T]} J_{t_0}(u; x_0)|_{M(x)=0}$. We can prove that

$$\sup\left\{|J_{t_0}^*(x_0)|:\ (t_0,x_0) \in [0,T] \times \Sigma\right\} < +\infty. \tag{5.10}$$

An estimate similar to (2.2) combined with (5.10) implies that for $(t, x) \in [0, T] \times \Sigma$, the optimal control $u(\cdot, t, x)$ for $(\mathrm{NQR})_{2,t}$ with initial state x is uniformly bounded in the norm of $U[t, T]$. This fact can be used to obtain the result of this lemma. ∎

THEOREM 5.4. *For any given $(t, x) \in [0, T] \times X$, there exists a unique solution of* (5.1). *The solution mapping $G(s, t, x)$ is continuously differentiable* *on* Ω, *and*

$$G_s(s,t,x) = A(s)G(s,t,x) - B(s)R^{-1}(s)B^*(s) \times \int_s^T E^*(\sigma,s)Q_x(\sigma,G(\sigma,t,x))\,d\sigma, \tag{5.11}$$

$$G_t(s,t,x) = \left\{\left(D_y F(G(\cdot,t,x),t,x)\right)^{-1} E(\cdot,t)\right\}(s) \times\left[-A(t)x + B(t)R^{-1}(t)B^*(t) \times \int_t^T E^*(\sigma,t)Q_x(\sigma,G(\sigma,t,x))\,d\sigma\right], \tag{5.12}$$

$$G_x(s,t,x) = \left\{\left(D_y F(G(\cdot,t,x),t,x)\right)^{-1} E(\cdot,t)\right\}(s), \tag{5.13}$$

where F *is given by* (5.7).

Proof. The existence and uniqueness have been proved by Lemma 5.2. Here we prove the continuity of $G(s,t,x)$ in $(s,t,x) \in \Omega$. Note that the implicit function theorem cannot be simply applied to this case because the space $X_c[t,T]$ associated with the mapping F varies with $t \in [0,T]$.

For a given $x_0 \in X$, we have

$$\lim_{\delta x \to 0} \|G(\cdot,t,x+\delta x) - G(\cdot,t,x)\|_{X_c[t,T]} = 0, \tag{5.14}$$

where the convergence is uniform with respect to $t \in [0,T]$. In fact, let $\Delta G(\cdot) = G(\cdot,t,x+\delta x) - G(\cdot,t,x)$. It satisfies

$$(I + \Gamma_t\Gamma_t^* Y_{t,\delta x})\Delta G(\cdot) = E(\cdot,t)\,\delta x, \tag{5.15}$$

where

$$(Y_{t,\delta x}\phi)(s) = \int_0^1 Q_{xx}(s,G(s,t,x) + \lambda\Delta G(s))\,d\lambda\,\phi(s), \qquad s \in [t,T].$$

From (5.3), $\|\Gamma_t\| = \|\Gamma_t^*\| \le \text{const}$, $0 \le t \le T$. From Lemma 5.3 and (A3), both $\|Y_{t,\delta x}\|_{L(X[t,T])}$ and $\|Y_{t,\delta x}\|_{L(X_c[t,T])}$ are uniformly bounded for $t \in [0,T]$ and sufficiently small $\|\delta x\|$. According to Lemma 5.1 and (5.4) we can assert that the convergence in (5.14) is uniform with respect to $t \in [0,T]$. Then, (5.14) can be used in turn to prove the continuity of $G(s,t,x)$ on Ω. The continuous differentiability of $G(s,t,x)$ along with (5.12) and (5.13) can be proved in the same way, and is omitted. Equation (5.11) is deduced from the direct differentiation of (5.6) in s. ∎

THEOREM 5.5. *Let* $G(s, t, x)$ *be the solution mapping of* (5.1). *Then,*

$$P(t,x) = \int_t^T E^*(\sigma,t) Q_x(\sigma, G(\sigma,t,x))\, d\sigma : [0,T] \times X \to X \quad (5.16)$$

is a normal solution of the quasi-Riccati equation (3.1) *associated with* $(\mathrm{NQR})_2$.

Proof. We can calculate $P_t(t, x)$ and $P_x(t, x)$ as in (4.8) and make use of (5.12) and (5.13) to verify that this $P(t, x)$ really satisfies the equation (3.1) with $M(x) \equiv 0$. Besides, we have

$$\begin{aligned} P_x(t,x) &= \int_t^T E^*(\sigma,t) Q_{xx}(\sigma, G(\sigma,t,x)) \\ &\qquad \times \left\{ \left(D_y F(G(\cdot,t,x),t,x) \right)^{-1} E(\cdot,t) \right\}(\sigma)\, d\sigma \\ &= V_t^* \sqrt{Q_{xx}(\cdot, G(\cdot))} \left(I + \sqrt{Q_{xx}(\cdot, G(\cdot))}\, \Gamma_t \Gamma_t^* \sqrt{Q_{xx}(\cdot, G(\cdot))} \right)^{-1} \\ &\qquad \times \sqrt{Q_{xx}(\cdot, G(\cdot))}\, V_t, \end{aligned} \quad (5.17)$$

where

$$\left(\sqrt{Q_{xx}(\cdot, G(\cdot))}\, \phi \right)(s) = \sqrt{Q_{xx}(s, G(s,t,x))}\, \phi(s), \qquad s \in [t,T],$$

and $V_t \in \mathbf{L}(X; \mathbf{X}[t, T])$ is defined by $(V_t x)(s) = E(s, t)x$. Hence $P_x(t, x) \in \mathbf{L}(X)$ is self-adjoint so that $P(t, x)$ shown by (5.16) is a gradient operator for each $t \in [0, T]$. Finally, by means of the semigroup relation

$$G(\sigma,t,x) = G(\sigma,s,G(s,t,x)), \qquad 0 \le t \le s \le \sigma \le T,\ x \in X \quad (5.18)$$

and (5.16) we can verify that $x(t) = G(t, 0, x_0)$ is a global solution to the Cauchy problem (3.2) corresponding to this $P(t, x)$. ∎

6. SYNTHESIS EQUATION FOR (NQR)

For the comprehensive problem (NQR), we consider the following synthesis equation which is a more complicated nonlinear integral equation,

$$\begin{aligned} y(s) &= E^*(T,s) M' \left(E(T,t)x - \int_t^T E(T,\eta) B(\eta) R^{-1}(\eta) B^*(\eta) y(\eta)\, d\eta \right) \\ &\quad + \int_s^T E^*(\sigma,s) \\ &\quad \times Q_x' \left(\sigma, E(\sigma,t)x - \int_t^\sigma E(\sigma,\eta) B(\eta) R^{-1}(\eta) B^*(\eta) y(\eta)\, d\eta \right) d\sigma, \\ &\qquad\qquad s \in [t,T];\ (t,x) \in [0,T] \times X. \end{aligned} \quad (6.1)$$

LEMMA 6.1. *Let $L_t \in \mathbf{L}(X[t,T])$ and $N_t \in \mathbf{L}(X[t,T]; X)$ be defined by*

$$(L_t\phi)(s) = \int_t^s E(s,\sigma)\phi(\sigma)\,d\sigma, \qquad s \in [t,T];\ \phi \in X[t,T].$$

$$N_t\phi = \int_t^T E(T,\sigma)\phi(\sigma)\,d\sigma; \qquad \phi \in X[t,T]. \tag{6.2}$$

(1) *Let $Z \in \mathbf{L}(X)$ and $Y \in \mathbf{L}(X[t,T])$ be any nonnegative operators. Then*

$$\Pi_t = I + (N_t^* Z N_t + L_t^* Y L_t) B(\cdot) R^{-1}(\cdot) B^*(\cdot) \in \mathbf{L}(X[t,T]) \tag{6.3}$$

is boundedly invertible and

$$\begin{aligned}\Pi_t^{-1} = I - (N_t^* Z N_t + L_t^* Y L_t)&\sqrt{B(\cdot)R^{-1}(\cdot)B^*(\cdot)}\\ &\times\Big\{I + \sqrt{B(\cdot)R^{-1}(\cdot)B^*(\cdot)}\,(N_t^* Z N_t + L_t^* Y L_t)\\ &\qquad\qquad \times\sqrt{B(\cdot)R^{-1}(\cdot)B^*(\cdot)}\Big\}^{-1}\\ &\times\sqrt{B(\cdot)R^{-1}(\cdot)B^*(\cdot)}\,.\end{aligned} \tag{6.4}$$

(2) *If $Y \in \mathbf{L}(X[t,T])$ is defined by*

$$(Y\phi)(s) = \Psi(s)\phi(s), \qquad s \in [t,T];\ \phi \in X[t,T],$$

where $\Psi(s) \in \mathbf{L}(X)$ is strongly continuous in s and is nonnegative self-adjoint, then (1) is valid, the corresponding $\Pi_t \in \mathbf{L}(X_c[t,T])$ is boundedly invertible, and Π_t^{-1} is still given by (6.4).

This result is similar to Lemma 5.1 and its proof is omitted.

THEOREM 6.2. *For any given $(t,x) \in [0,T] \times X$, there exists a unique solution $y(\cdot) \in X_c[t,T]$ of (6.1). Denote this solution by a mapping $y(s) = W(s,t,x)$. Then, $W(s,t,x)\colon \Omega \to X$ is continuous in $(s,t,x) \in \Omega$.*

Proof. By Corollary 2.3, let the unique optimal process of $(\mathrm{NQR})_t$ with the initial state x be denoted by $\{\hat{u}(\cdot;t,x), \hat{x}(\cdot;t,x)\}$. From (2.4) and (2.5) we can verify that

$$y(s) = E^*(T,s)M'(\hat{x}(T;t,x)) + \int_s^T E^*(\sigma,s)Q_x(\sigma,\hat{x}(\sigma;t,x))\,d\sigma, \qquad s \in [t,T], \tag{6.5}$$

is exactly a solution of Eq. (6.1) in $\mathcal{X}_c[t,T]$. Define a mapping F_t: $\mathcal{X}_c[t,T] \times X \to \mathcal{X}_c[t,T]$ by

$$
\begin{aligned}
&F_t(y(\cdot),x)(s)\\
&\quad = y(s) - E^*(T,s)\\
&\qquad \times M'\left(E(T,t)x - \int_t^T E(T,\eta)B(\eta)R^{-1}(\eta)B^*(\eta)y(\eta)\,d\eta\right)\\
&\qquad - \int_s^T E^*(\sigma,s)Q_x\Bigg(\sigma, E(\sigma,t)x\\
&\qquad\qquad - \int_t^\sigma E(\sigma,\eta)B(\eta)R^{-1}(\eta)B^*(\eta)y(\eta)\,d\eta\Bigg)\,d\sigma,\\
&\qquad\qquad s\in[t,T]. \qquad (6.6)
\end{aligned}
$$

We can calculate that

$$D_yF_t(y(\cdot),x) = I + \left(N_t^*Z_yN_t + L_t^*Y_yL_t\right)B(\cdot)R^{-1}(\cdot)B^*(\cdot), \quad (6.7)$$

where L_t and N_t are shown by (6.2), and

$$Z_y = M''\left(E(T,t)x - \int_t^T E(T,\xi)B(\xi)R^{-1}(\xi)B^*(\xi)y(\xi)\,d\xi\right), \quad (6.8)$$

$$
\begin{aligned}
&(Y_y\phi)(s)\\
&\quad = Q_{xx}\left(s, E(s,t)x - \int_t^s E(s,\xi)B(\xi)R^{-1}(\xi)B^*(\xi)y(\xi)\,d\xi\right)\phi(s),\\
&\qquad s\in[t,T];\ \phi\in\mathcal{X}[t,T]. \qquad (6.9)
\end{aligned}
$$

By (A2), (A3), and Lemma 6.1, $D_yF_t(y(\cdot),x) \in \mathcal{L}(\mathcal{X}_c[t,T])$ is boundedly invertible. This implies the uniqueness.

Finally, the continuity of the solution mapping $W(s,t,x)$ can be proved based on the following two facts: first,

$$\sup_{(s,t,x)\in\Omega_\Sigma} \|W(s,t,x)\|_X < +\infty, \quad (6.10)$$

where $\Omega_\Sigma = \{(s,t,x) | 0 \le t \le s \le T,\ x\in\Sigma\}$ and Σ is any fixed bounded subset in X; and second, for each $x\in X$, the convergence

$$\lim_{\delta x\to 0} \|W(\cdot,t,x,+\delta x) - W(\cdot,t,x)\|_{\mathcal{X}_c[t,T]} = 0 \quad (6.11)$$

is uniform with respect to $t\in[0,T]$. ∎

THEOREM 6.3. *$W(s,t,x)$ is continuously differentiable in s, t, and x in Ω, and*

$$W_s(s,t,x) = -A^*(s)W(s,t,x) - Q_x(s,\Psi(s,t,x)), \tag{6.12}$$

$$\begin{aligned} W_t(s,t,x) = \left(D_y F_t(W(\cdot,t,x),x)\right)^{-1} &\Big\{E^*(T,\cdot)M''(\Psi(T,t,x))E(T,t) \\ &+ \int_\cdot^T E^*(\sigma,\cdot)Q_{xx}(\sigma,\Psi(\sigma,t,x))E(\sigma,t)\,d\sigma\Big\}(s) \\ &\times\left[-A(t)x + B(t)R^{-1}(t)B^*(t)W(t,t,x)\right], \end{aligned} \tag{6.13}$$

$$\begin{aligned} W_x(s,t,x) = \left(D_y F_t(W(\cdot,t,x),x)\right)^{-1} &\Big\{E^*(T,\cdot)M''(\Psi(T,t,x))E(T,t) \\ &+ \int_\cdot^T E^*(\sigma,\cdot)Q_{xx}(\sigma,\Psi(\sigma,t,x))E(\sigma,t)\,d\sigma\Big\}(s), \end{aligned} \tag{6.14}$$

where F_t is shown in (6.6), *and*

$$\Psi(s,t,x) = E(s,t)x - \int_t^s E(s,\eta)B(\eta)R^{-1}(\eta)B^*(\eta)W(\eta,t,x)\,d\eta, \quad 0 \le t \le s \le T,\ x \in X. \tag{6.15}$$

Proof. Equation (6.12) is obtained by direct differentiation of (6.1) with respect to s. From

$$W_x(s,t,x) = -\left[\left(D_y F_t(W(\cdot,t,x),x)\right)\right]^{-1}\left(D_x F_t(W(\cdot,t,x),x)\right) \tag{6.16}$$

and

$$\begin{aligned} D_x F_t(W(\cdot,t,x),x) = &-E^*(T,\cdot)M''(\Psi(T,t,x))E(T,t) \\ &- \int_\cdot^T E^*(\sigma,\cdot)Q_{xx}(\sigma,\Psi(\sigma,t,x))E(\sigma,t)\,d\sigma, \end{aligned} \tag{6.17}$$

we can verify (6.14). Finally, (6.13) can be deduced by a tedious but straightforward calculation of the limit of the related difference quotient. Here we omit the details. ∎

THEOREM 6.4. *Let $W(s,t,x)$ be the solution mapping of* (6.1). *Then,*

$$P(t,x) = W(t,t,x)\colon [0,T] \times X \to X \tag{6.18}$$

is a normal solution of the quasi-Riccati equation (3.1) *for* (NQR).

Proof. From (6.12), (6.13), and (6.14), we have

$$
\begin{aligned}
P_t(t,x) &= [W_s(s,t,x) + W_t(s,t,x)]|_{s=t} \\
&= -A^*(t)P(t,x) - Q_x(t,x) + \big(D_y F_t(W(\cdot,t,x),x)\big)^{-1} \\
&\quad \times \Big\{ E^*(T,\cdot) M''(\Psi(T,t,x)) E(T,t) \\
&\qquad + \int_{\cdot}^{T} E^*(\sigma,\cdot) Q_{xx}(\sigma,\Psi(\sigma,t,x)) E(\sigma,t)\, d\sigma \Big\}(t) \\
&\quad \times \big[-A(t)x + B(t)R^{-1}(t)B^*(t)P(t,x)\big],
\end{aligned}
\tag{6.19}
$$

and

$$
\begin{aligned}
P_x(t,x) = \big(D_y F_y(W(\cdot,t,x),x)\big)^{-1} \Big\{ & E^*(T,\cdot) M''(\Psi(T,t,x)) E(T,t) \\
& + \int_{\cdot}^{T} E^*(\sigma,\cdot) Q_{xx}(\sigma,\Psi(\sigma,t,x)) E(\sigma,t)\, d\sigma \Big\}(t).
\end{aligned}
\tag{6.20}
$$

Equations (6.19) and (6.20) imply that $P(t,x)$ given by (6.18) satisfies Eq. (3.1) and $P(T,x) = W(T,T,x) = M'(x)$, for $x \in X$.

Next, for each fixed $t \in [0,T]$, $P_x(t,x) \in \mathsf{L}(X)$ is self-adjoint. In fact,

$$
\begin{aligned}
P_x(t,x) &= W_x(t,t,x) \\
&= E^*(T,t) M''(\Psi(T,t,x)) E(T,t) \\
&\quad + \int_t^T E^*(\sigma,t) Q_{xx}(\sigma,\Psi(\sigma,t,x)) E(\sigma,t)\, d\sigma \\
&\quad - S_t^t \sqrt{BR^{-1}B^*} \Big(I + \sqrt{BR^{-1}B^*}\, S_t \sqrt{BR^{-1}B^*} \Big)^{-1} \sqrt{BR^{-1}B^*} \, (S_t^t)^*,
\end{aligned}
\tag{6.21}
$$

where

$$
\begin{gathered}
S_t = N_t^* Z_\Psi N_t + L_t^* Y_\Psi L_t \in \mathsf{L}(\mathsf{X}[t,T]), \\
S_t^t \phi = (S_t \phi)(t), \qquad \phi \in \mathsf{X}[t,T]; \qquad S_t^t \in \mathsf{L}(\mathsf{X}[t,T]; X), \\
Z_\Psi = M''(\Psi(T,t,x)), \\
(Y_\Psi \phi)(\sigma) = Q_{xx}(\sigma,\Psi(\sigma,t,x))\phi(\sigma), \qquad \sigma \in [t,T],\ \phi \in \mathsf{X}[t,T],
\end{gathered}
\tag{6.22}
$$

and $\sqrt{BR^{-1}B^*} \in \mathsf{L}(\mathsf{X}[t,T])$ is a multiplication operator. Hence $P(t,\cdot)$ is a gradient operator for each $t \in [0,T]$.

Finally we can use Lemma 6.1 to prove that

$$W(t,0,x) = W(t,t,\Psi(t,0,x)), \qquad (t,x) \in [0,T] \times X. \quad (6.23)$$

Then we can verify that

$$x(t) = \Psi(t,0,x_0), \qquad t \in [0,T], \quad (6.24)$$

is a global solution of the Cauchy problem (3.2) corresponding to this $P(t,x)$. In fact, by (6.15), (6.23), and (6.18), we obtain

$$\begin{aligned} x(t) &= E(t,0)x_0 - \int_0^t E(t,\eta)B(\eta)R^{-1}(\eta)B^*(\eta)W(\eta,0,x_0)\,d\eta \\ &= E(t,0)x_0 - \int_0^t E(t,\eta)B(\eta)R^{-1}(\eta)B^*(\eta)P(\eta,x(\eta))\,d\eta. \end{aligned} \quad (6.25)$$

Therefore, this $P(t,x)$ is a normal solution of (3.1) for (NQR). ▮

7. REMARKS AND EXAMPLE

Remark 1. If $M(x)$ and $Q(t,x)$ in (1.2) reduce to quadratic forms $M(x) = \frac{1}{2}\langle \hat{M}x, x\rangle$ and $Q(t,x) = \frac{1}{2}\langle \hat{Q}(t)x, x\rangle$ where $\hat{M} \geq 0$, $\hat{Q}(t) \geq 0$ are self-adjoint, and $\hat{Q}(t)$ is continuous, then we can verify that $P(t,x)$ given by (4.7), (5.16), and (6.18) in three cases respectively become linear in x, i.e., $P(t,x) = \hat{P}(t)x$, where $\hat{P}(t) \in L(X)$ will exactly be the continuous and nonnegative symmetric solution of the following Riccati matrix equation:

$$\begin{cases} \dfrac{d}{dt}\hat{P}(t) + \hat{P}(t)A(t) + A^*(t)\hat{P}(t) - \hat{P}(t)B(t)R^{-1}(t)B^*(t)\hat{P}(t) \\ \qquad + \hat{Q}(t) = 0, \qquad 0 \leq t \leq T, \\ \hat{P}(T) = \hat{M}. \end{cases} \quad (7.1)$$

Remark 2. The assumptions (A1) and (A3) can be weakened to some extent. If $A(t)$ and $B(t)$ are assumed to be bounded and measurable, and $Q(t,x)$ along with Q_x and Q_{xx} are measurable and locally bounded in (t,x), then the results still hold after some technical modification. Convexity, however, is required.

EXAMPLE. Consider a simple nonquadratic optimal control problem for a scalar linear system,

$$\frac{dx}{dt} = a(t)x + b(t)u(t), \qquad x(0) = x_0 \in \mathbf{R}$$

$$\min_{u \in L^2(0,T)} \left\{ J(u) = m|x(T)|^4 + \frac{1}{2}\int_0^T r(t)|u(t)|^2\, dt \right\}. \tag{7.2}$$

Suppose that $a(t)$, $b(t)$, and $r(t)$ are continuous, $r(t) \geq \delta \geq 0$, and $m > 0$.

The synthesis equation (4.1) for this problem becomes

$$\beta(t)y^3 + y = \gamma(t)x, \tag{7.3}$$

where

$$\beta(t) = 4m\int_t^T (b^2(s)/r(s))\exp\left(2\int_s^T a(\sigma)\, d\sigma\right) ds \qquad \text{and}$$

$$\gamma(t) = \exp\left[\int_t^T a(\sigma)\, d\sigma\right].$$

For $t \in [\,0, T)$, let

$$\Delta(t,x) = \left[\frac{-\gamma(t)x}{2\beta(t)}\right]^2 + \left[\frac{1}{3\beta(t)}\right]^3.$$

According to Cardan's formula, the unique real root of Eq. (7.3) is given by

$$y = H(t,x)$$
$$= \begin{cases} \sqrt[3]{\dfrac{1}{2\beta(t)}\gamma(t)x + \sqrt{\Delta(t,x)}} - \sqrt{-\dfrac{1}{2\beta(t)}\gamma(t)x + \sqrt{\Delta(t,x)}} \\ \qquad \text{if } 0 < t < T; \\ x, \qquad \text{if } t = T. \end{cases} \tag{7.4}$$

Substitution of (7.4) into (4.7) gives rise to the nonlinear feedback operator

$$P(t,x) = 4m \exp\left[\int_t^T a(\sigma)\, d\sigma\right][H(t,x)]^3 \tag{7.5}$$

and the nonlinear closed-loop optimal control

$$u(t) = -\frac{b(t)}{r(t)}P(t,x(t)), \qquad t \in [0,T]. \tag{7.6}$$

8. CONCLUSIONS

In this paper, the well-known finite dimensional time-variant linear-quadratic control theory is generalized to nonquadratic optimal control problems. On the other hand, this work is a generalization of the author's previous work [5, 6] on the time-invariant problems. We summarize the main results as a closed-loop theorem:

THEOREM 8.1. *For any given* $x_0 \in X$, *there exists a unique optimal control for* (NQR), *given by*

$$u(t) = -R^{-1}(t)B^*(t)P(t, x(t)), \qquad t \in [0, T],$$

where the feedback operator $P(t, x)$ *is a normal solution of the quasi-Riccati equation* (3.1) *and* $P(t, x)$ *is given respectively by*

$$P(t, x) = E^*(T, t)M'(H(t, x)), \qquad \text{for } (\mathrm{NQR})_1$$

$$P(t, x) = \int_t^T E^*(\sigma, t)Q_x(\sigma, G(\sigma, t, x))\, d\sigma, \qquad \text{for } (\mathrm{NQR})_2$$

$$P(t, x) = W(t, t, x), \qquad \text{for } (\mathrm{NQR})$$

in which $H(t, x)$, $G(s, t, x)$, *and* $W(s, t, x)$ *are the solution mappings of the synthesis equations* (4.1), (5.1), *and* (6.1), *respectively*.

In contrast to the linear state feedback and the associated Riccati equations for the quadratic performance, here it is proved that the nonquadratic closed-loop optimal control is a nonlinear state feedback and that the nonlinear feedback operator is a normal solution of a quasi-Riccati equation. The approach of expressing $P(t, x)$ by solutions of the described nonlinear time-dependent algebraic and integral equations will be advantageous for numerical computation in applications.

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